以下の問題では積分定数を $C$ としなさい．（必要に応じて $C'$ を用いても良い.）

# 1 不定積分/基本

次の不定積分を解け.

(1) $\displaystyle\int 3\sqrt{x}\,\mathrm{d}x = \underline{\underline{2\sqrt{x^3}+C(=2x\sqrt{x}+C)}}$

(2) $\displaystyle\int \frac{x^3-2x+1}{x^2}\,\mathrm{d}x = \underline{\underline{\frac{1}{2}x^2-2\log|x|-\frac{1}{x}+C}}$

(3) $\displaystyle\int \frac{(\sqrt{x}-2)^2}{\sqrt{x}}\,\mathrm{d}x = \underline{\underline{\frac{2}{3}x\sqrt{x}-4x+8\sqrt{x}+C}}$

(4) $\displaystyle\int \frac{(\sqrt[3]{x}-1)^3}{x}\,\mathrm{d}x = \underline{\underline{x-\frac{9}{2}\sqrt[3]{x^2}+9\sqrt[3]{x}-\log|x|+C}}$

(5) $\displaystyle\int e^{x+2}\,\mathrm{d}x = \underline{\underline{e^{x+2}+C}}$

(6) $\displaystyle\int 5^{x-1}\,\mathrm{d}x = \underline{\underline{\frac{5^{x-1}}{\log 5}+C}}$

(7) $\displaystyle\int \frac{x}{\sqrt[3]{x+1}-1}\,\mathrm{d}x$

$$
\begin{aligned}
(\text{与式}) &= \int \frac{x\left\{\left(\sqrt[3]{x+1}\right)^2+\sqrt[3]{x+1}+1\right\}}{\left(\sqrt[3]{x+1}-1\right)\left\{\left(\sqrt[3]{x+1}\right)^2+\sqrt[3]{x+1}+1\right\}}\,\mathrm{d}x \\
&= \int \frac{x\left\{\left(\sqrt[3]{x+1}\right)^2+\sqrt[3]{x+1}+1\right\}}{x}\,\mathrm{d}x \\
&= \int \left\{(x+1)^{\frac{2}{3}}+(x+1)^{\frac{1}{3}}+1\right\}\mathrm{d}x \\
&= \underline{\underline{\frac{3}{5}(x+1)^{\frac{5}{3}}+\frac{3}{4}(x+1)^{\frac{4}{3}}+x+C}}
\end{aligned}
$$

◀ 分母の有理化をする.

# 2 不定積分/三角関数

次の不定積分を解け.

(1) $\displaystyle\int (7\cos x-5\sin x)\,\mathrm{d}x$

$= \underline{\underline{7\sin x+5\cos x+C}}$

(2) $\displaystyle\int \frac{7}{\sin^2 x}\,\mathrm{d}x$

$= \underline{\underline{-\dfrac{7}{\tan x}+C}}$

(3) $\displaystyle\int \frac{5}{\cos^2 x}\,\mathrm{d}x$

$= \underline{\underline{5\tan x+C}}$

(4) $\displaystyle\int \cos 2x \sin 5x \, \mathrm{d}x$

$$(\text{与式}) = \int \sin 5x \cos 2x \, \mathrm{d}x = \frac{1}{2}\int (\sin 7x + \sin 3x)\ \mathrm{d}x$$
$$= \underline{\underline{-\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + C}}$$

◀ 和積の公式.

(5) $\displaystyle\int \cos x \cos 4x \, \mathrm{d}x$

$$(\text{与式}) = \int \cos 4x \cos x \, \mathrm{d}x = \frac{1}{2}\int (\cos 5x + \cos 3x)\ \mathrm{d}x$$
$$= \underline{\underline{\frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + C}}$$

(6) $\displaystyle\int \cos 4x \sin 3x \, \mathrm{d}x$

$$(\text{与式}) = \frac{1}{2}\int (\sin 7x - \sin x)\ \mathrm{d}x$$
$$= \underline{\underline{-\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + C}}$$

(7) $\displaystyle\int \sin 3x \sin 5x \, \mathrm{d}x$

$$(\text{与式}) = \int \sin 5x \sin 3x \, \mathrm{d}x = -\frac{1}{2}\int (\cos 8x - \cos 2x)\ \mathrm{d}x$$
$$= \underline{\underline{-\frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C}}$$

(8) $\displaystyle\int \sin 2x \cos x \, \mathrm{d}x$

$$(\text{与式}) = \frac{1}{2}\int (\sin 3x + \sin x)\ \mathrm{d}x$$
$$= \underline{\underline{-\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C}}$$

(9) $\displaystyle\int \cos x \sin 2x \, \mathrm{d}x$

$$(\text{与式}) = \int \sin 2x \cos x \, \mathrm{d}x = \frac{1}{2}\int (\sin 3x + \sin x)\ \mathrm{d}x$$
$$= \underline{\underline{-\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C}}$$

(10) $\displaystyle\int \cos^2 \frac{x}{2}\,\mathrm{d}x$

$$
\begin{aligned}
(\text{与式}) &= \int \frac{1+\cos x}{2}\,\mathrm{d}x \\
&= \int \left(\frac{1}{2}+\frac{\cos x}{2}\right)\mathrm{d}x \\
&= \underline{\underline{\frac{x}{2}+\frac{1}{2}\sin x + C}}
\end{aligned}
$$

◀ 半角の公式.

## 3　不定積分/部分分数分解

次の不定積分を解け.

(1) $\displaystyle\int \frac{1}{(x+1)(x+2)}\,\mathrm{d}x = \int \left(\frac{1}{x+1}-\frac{1}{x+2}\right)\mathrm{d}x$
$= \underline{\underline{\log|x+1|-\log|x+2|+C}}$

(2) $\displaystyle\int \frac{5x+3}{(x-1)(x+3)}\,\mathrm{d}x = \int \left(\frac{2}{x-1}+\frac{3}{x+3}\right)\mathrm{d}x$
$= \underline{\underline{2\log|x-1|+3\log|x+3|+C}}$

(3) $\displaystyle\int \frac{-7x^2-13x-4}{x(x+1)(x+2)}\,\mathrm{d}x = \int \left(-\frac{2}{x}+\frac{-2}{x+1}+\frac{-3}{x+2}\right)\mathrm{d}x$
$= \underline{\underline{-2\log|x|-2\log|x+1|-3\log|x+2|+C}}$

(4) $\displaystyle\int \frac{-2x^2-12x-19}{x^3+7x^2+16x+12}\,\mathrm{d}x = \int \frac{-2x^2-12x-19}{(x+2)^2(x+3)}\,\mathrm{d}x$
$\displaystyle= \int \left\{\frac{-3}{(x+2)^2}+\frac{-1}{x+2}+\frac{-1}{x+3}\right\}\mathrm{d}x$
$\displaystyle= \underline{\underline{\frac{3}{x+2}-\log|x+2|-\log|x+3|+C}}$

◀ 2 次式があるときの部分分数分解は注意して行う.

## 4　不定積分/置換積分（基本）

次の不定積分を解け.

(1) $\displaystyle\int (3x+1)^3\,\mathrm{d}x$

$3x+1=t$ とおくと $3\cdot\mathrm{d}x = 1\cdot\mathrm{d}t \Leftrightarrow \mathrm{d}x = \dfrac{1}{3}\,\mathrm{d}t$

$$(\text{与式}) = \int t^3\frac{1}{3}\,\mathrm{d}t = \frac{1}{12}t^4 + C = \underline{\underline{\frac{1}{12}(3x+1)^4+C}}$$

◀ 置換するときは, できるだけ大きなカタマリで置換するのが基本.

(2) $\displaystyle\int (x^3+1)^2\cdot 3x^2\,\mathrm{d}x$

$x^3+1=t$ とおくと $3x^2\cdot\mathrm{d}x = \mathrm{d}t$

$$\int (x^3+1)^2\cdot \underset{\sim\sim\sim}{3x^2\,\mathrm{d}x} = \int t^2\,\underset{\sim}{\mathrm{d}t} = \underline{\underline{\frac{1}{3}(x^3+1)^3+C}}$$

(3) $\displaystyle\int \frac{6x}{3x^2-5}\,\mathrm{d}x$

$3x^2-5=t$ とおくと $6x\cdot\mathrm{d}x=\mathrm{d}t$

$$\int \frac{6x}{3x^2-5}\,\mathrm{d}x = \int \frac{1}{t}\,\mathrm{d}t = \underline{\underline{\log|3x^2-5|+C}}$$

(4) $\displaystyle\int \frac{x}{(x^2+4)^2}\,\mathrm{d}x$

$t=x^2+4$ とおくと $\mathrm{d}t=2x\,\mathrm{d}x$ なので

$$(\text{与式}) = \int \frac{\mathrm{d}t}{2t^2} = -\frac{1}{2t}+C = \underline{\underline{-\frac{1}{2(x^2+4)}+C}}$$

(5) $\displaystyle\int (x+2)(x^2+4x+1)^3\,\mathrm{d}x$

$t=x^2+4x+1$ とおくと $\mathrm{d}t=(2x+4)\,\mathrm{d}x=2(x+2)\,\mathrm{d}x$ なので

$$(\text{与式}) = \int \frac{t^3}{2}\,\mathrm{d}t = \frac{t^4}{8}+C = \underline{\underline{\frac{1}{8}(x^2+4x+1)^4+C}}$$

(6) $\displaystyle\int \frac{1}{x^2}\cdot\left(1+\frac{2}{x}\right)^2\mathrm{d}x$

$t=1+\dfrac{2}{x}$ とおくと， $\mathrm{d}t=-\dfrac{2}{x^2}\,\mathrm{d}x$ だから

$$(\text{与式}) = \int \frac{t^2}{2}(-\mathrm{d}t) = -\frac{t^3}{6}+C = \underline{\underline{-\frac{1}{6}\left(1+\frac{2}{x}\right)^3+C}}$$

(7) $\displaystyle\int x\sqrt{1-x}\,\mathrm{d}x$

$t=\sqrt{1-x}$ とおくと $\mathrm{d}t=-\dfrac{\mathrm{d}x}{2\sqrt{1-x}}=-\dfrac{\mathrm{d}x}{2t}$， $x=1-t^2$ なので

◀ $t=1-x$ としても（やってみるとわかるが）うまくいかない．このようなときはルートごと置換するとうまくいくことが多い．

$$\begin{aligned}(\text{与式}) &= \int \left\{(1-t^2)\cdot t\cdot(-2t\,\mathrm{d}t)\right\} = \int (2t^4-2t^2)\,\mathrm{d}t \\ &= \frac{2t^5}{5}-\frac{2t^3}{3}+C \\ &= \frac{2t^3}{15}(3t^2-5)+C \\ &= \frac{2}{15}(1-x)\sqrt{1-x}\{3(1-x)-5\}+C \\ &= \underline{\underline{-\frac{2}{15}(3x+2)(1-x)\sqrt{1-x}+C}}\end{aligned}$$

(8) $\displaystyle\int x\sqrt{1-x^2}\,\mathrm{d}x$

$t=1-x^2$ とおくと $\mathrm{d}t=-2x\,\mathrm{d}x$ なので

◀ これはルートの中身を置換するほうが簡単にできるパターン．暗記するよりは, 試してみてだめだったら方針転換すると良い．

$$(\text{与式}) = \int\left(-\frac{1}{2}\right)\sqrt{t}\,\mathrm{d}t = -\frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}}+C$$

$$= \underline{\underline{-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C}}$$

(9) $\displaystyle\int \frac{3x-1}{\sqrt{x+1}}\,\mathrm{d}x$

$t=\sqrt{x+1}$ とおくと $\mathrm{d}t=\dfrac{\mathrm{d}x}{2\sqrt{x+1}}$ なので $(x=t^2-1)$

$$
\begin{aligned}
(\text{与式}) &= \int (3x-1)\cdot 2\cdot \frac{\mathrm{d}x}{2\sqrt{x+1}} = \int 2(3t^2-4)\,\mathrm{d}t \\
&= 2(t^3-4t)+C = 2t(t^2-4)+C \\
&= 2((x+1)-4)\sqrt{x+1}+C = \underline{\underline{2(x-3)\sqrt{x+1}+C}}
\end{aligned}
$$

(10) $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,\mathrm{d}x$

$t=\sqrt{1-x^2}$ とおくと $\mathrm{d}t=-\dfrac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ なので $(x^2=1-t^2)$

$$
\begin{aligned}
\frac{\mathrm{d}x}{x} &= -\frac{\sqrt{1-x^2}x^2\,\mathrm{d}t}{=} - \frac{1-t^2}{t}\,\mathrm{d}t = \frac{t^2-1}{t}\,\mathrm{d}t \\
(\text{与式}) &= \int t\cdot\frac{t}{t^2-1}\,\mathrm{d}t = \int\left(1+\frac{1}{t^2-1}\right)\mathrm{d}t \\
&= \int\left(1+\frac{1}{2(t-1)}-\frac{1}{2(t+1)}\right)\mathrm{d}t \\
&= t+\frac{1}{2}\left(\log|t-1|-\log|t+1|\right)+C \\
&= t+\log\sqrt{\left|\frac{t-1}{t+1}\right|}+C \\
&= \underline{\underline{\sqrt{1-x^2}+\log\frac{1-\sqrt{1-x^2}}{|x|}+C}}
\end{aligned}
$$

(11) $\displaystyle\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}\,\mathrm{d}x$

$t=\sqrt[4]{x^3}+1$ とおくと $\mathrm{d}t=\dfrac{3}{4\sqrt[4]{x}}\,\mathrm{d}x$ なので

$$(\text{与式}) = \int \frac{\sqrt[4]{x^2}}{(\sqrt[4]{x^3}+1)} \cdot \frac{4}{3}\sqrt[4]{x} \cdot \underset{\sim\sim\sim}{\frac{3}{4\sqrt[4]{x}}}\,\mathrm{d}x$$

◀ 波線部を得るための変形．

$$= \frac{4}{3}\int \frac{\sqrt[4]{x^3}}{(\sqrt[4]{x^3}+1)} \cdot \underset{\sim\sim\sim}{\frac{3}{4\sqrt[4]{x}}}\,\mathrm{d}x$$

$$= \frac{4}{3}\int \frac{t-1}{t}\,\underset{\sim}{\mathrm{d}t}$$

◀ 文字を置換するときは，一部ずつではなくまとめて置換するほうがよい．

$$= \frac{4}{3}\int \left(1-\frac{1}{t}\right)\mathrm{d}t$$

$$= \frac{4}{3}t - \frac{4}{3}\log|t| + C$$

$$= \underline{\underline{\frac{4}{3}(\sqrt[4]{x^3}+1) - \frac{4}{3}\log\left(\sqrt[4]{x^3}+1\right) + C}}$$

◀ これでも正解．

$$= \underline{\underline{\frac{4}{3}\sqrt[4]{x^3} - \frac{4}{3}\log\left(\sqrt[4]{x^3}+1\right) + C'}}$$

◀ 展開して出てくる $\frac{4}{3}$ は定数なので，積分定数にまとめてしまうことが多い．

# 5 不定積分/置換積分（三角・指数）

次の不定積分を解け．

(1) $\displaystyle\int \frac{\mathrm{d}x}{(1+\tan x)\cos^2 x}$

$t = 1+\tan x$ とおくと $\mathrm{d}t = \dfrac{\mathrm{d}x}{\cos^2 x}$ なので

$(\text{与式}) = \displaystyle\int \frac{\mathrm{d}t}{t} = \log|t| + C = \underline{\underline{\log|1+\tan x| + C}}$

◀ $t = \tan x$ でもよいが，できるだけ大きなカタマリで置換する方が計算が楽．

(2) $\displaystyle\int \frac{\tan x}{\cos^3 x}\,\mathrm{d}x$

$$(\text{与式}) = \int \frac{\sin x}{\cos^4 x}\,\mathrm{d}x$$

◀ $\sin x\,\mathrm{d}x$ というカタマリが出てくると置換積分できそう，と気付けるようになるのが目標．

$t = \cos x$ とおくと $\mathrm{d}t = -\sin x\,\mathrm{d}x$ なので

$$= -\int \frac{\mathrm{d}t}{t^4} = \frac{1}{3t^3} + C$$

$$= \underline{\underline{\frac{1}{3\cos^3 x} + C}}$$

(3) $\displaystyle\int (2x+1)e^{x^2+x+5}\,\mathrm{d}x$

$t = x^2+x+5$ とおくと $\mathrm{d}t = (2x+1)\,\mathrm{d}x$ なので

$(\text{与式}) = \displaystyle\int e^t\,\mathrm{d}t = e^t + C = \underline{\underline{e^{x^2+x+5} + C}}$

別解

$(x^2+x+5)'=2x+1$ より，(与式) $=\underline{\underline{e^{x^2+x+5}+C}}$

◀ これができるようになると計算がかなり速くなる．

(4) $\displaystyle\int \frac{\log x}{2x}\,\mathrm{d}x$

$t=\log x$ とおくと $\mathrm{d}t=\dfrac{\mathrm{d}x}{x}$ なので

(与式) $\displaystyle =\int \frac{t}{2}\,\mathrm{d}t=\frac{t^2}{4}+C=\underline{\underline{\frac{(\log x)^2}{4}+C}}$

(5) $\displaystyle\int \cos^3 x\sin^2 x\,\mathrm{d}x$

$$\begin{aligned}(\text{与式})&=\int \cos x(1-\sin^2 x)\sin^2 x\,\mathrm{d}x\\&=\int \cos x(\sin^2 x-\sin^4 x)\,\mathrm{d}x\end{aligned}$$

◀ $\cos x\,\mathrm{d}x$ のカタマリ．

$t=\sin x$ とおくと $\mathrm{d}t=\cos x\,\mathrm{d}x$ なので

$$\begin{aligned}&=\int (t^2-t^4)\,\mathrm{d}t=\frac{t^3}{3}-\frac{t^5}{5}+C\\&=\underline{\underline{\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C}}\end{aligned}$$

(6) $\displaystyle\int \frac{\cos x}{\sin x(\sin x+1)}\,\mathrm{d}x$

$t=\sin x$ とおくと $\mathrm{d}t=\cos x\,\mathrm{d}x$ なので

◀ $\cos x\,\mathrm{d}x$ のカタマリ．

$$\begin{aligned}(\text{与式})&=\int \frac{\mathrm{d}t}{t(t+1)}=\int\left(\frac{1}{t}-\frac{1}{t+1}\right)\mathrm{d}t\\&=\log\left|\frac{t}{t+1}\right|+C\\&=\underline{\underline{\log\left|\frac{\sin x}{\sin x+1}\right|+C}}\end{aligned}$$

(7) $\displaystyle\int \frac{\sin 2x}{1+\sin x}\,\mathrm{d}x$

$$(\text{与式})=\int \frac{2\sin x\cos x}{1+\sin x}\,\mathrm{d}x$$

$t=\sin x$ とおくと $\mathrm{d}t=\cos x\,\mathrm{d}x$ なので

$$\begin{aligned}&=\int \frac{2t}{1+t}\,\mathrm{d}t=2\int\left(1-\frac{1}{1+t}\right)\mathrm{d}t\\&=2t-2\log|1+t|+C\\&=\underline{\underline{2\sin x-2\log(1+\sin x)+C}}\end{aligned}$$

(8) $\displaystyle\int \frac{e^{3x}}{e^x-1}\,\mathrm{d}x$

$t=e^x$ とおくと $\mathrm{d}t=e^x\,\mathrm{d}x$ なので

$$(与式)=\int\frac{t^2}{t-1}\,\mathrm{d}t=\int\left(t+1+\frac{1}{t-1}\right)\mathrm{d}t$$
$$=\frac{1}{2}t^2+t+\log|t-1|+C$$
$$=\underline{\underline{\frac{1}{2}e^{2x}+e^x+\log|e^x-1|+C}}$$

◀ 分母の次数の方が大きいので割り算を行う．

(9) $\displaystyle\int \frac{e^x}{e^x-e^{-x}}\,\mathrm{d}x$

$$(与式)=\int\frac{e^{2x}}{e^{2x}-1}\,\mathrm{d}x$$

$t=e^{2x}-1$ とおくと $\mathrm{d}t=2e^{2x}\,\mathrm{d}x$ なので

◀ $t=e^{2x}$ でもよい．

$$=\frac{1}{2}\int\frac{\mathrm{d}t}{t}=\frac{1}{2}\log|t|+C \qquad =\underline{\underline{\frac{1}{2}\log|e^{2x}-1|+C}}$$

(10) $\displaystyle\int \frac{\mathrm{d}x}{1+\cos x}$

$$(与式)=\int\frac{1-\cos x}{1-\cos^2 x}\,\mathrm{d}x=\int\left(\frac{1}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)\mathrm{d}x$$
$$=-\frac{1}{\tan x}-\int\frac{\cos x}{\sin^2 x}\,\mathrm{d}x$$

$t=\sin x$ とおくと $\mathrm{d}t=\cos x\,\mathrm{d}x$ なので

$$=-\frac{1}{\tan x}-\int\frac{\mathrm{d}t}{t^2}=-\frac{1}{\tan x}+\frac{1}{t}+C$$
$$=\underline{\underline{-\frac{1}{\tan x}+\frac{1}{\sin x}+C}}$$

(11) $\displaystyle\int \frac{\mathrm{d}x}{1+\sin x}$

$$(与式)=\int\frac{1-\sin x}{1-\sin^2 x}\,\mathrm{d}x=\int\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)\mathrm{d}x$$
$$=\tan x-\int\frac{\sin x}{\cos^2 x}\,\mathrm{d}x$$

$t = \cos x$ とおくと $\mathrm{d}t = -\sin x\,\mathrm{d}x$ なので

$$= \tan x + \int \frac{\mathrm{d}t}{t^2}\,\mathrm{d}t = \tan x - \frac{1}{t} + C$$
$$= \underline{\underline{\tan x - \frac{1}{\cos x} + C}}$$

(12) $\displaystyle\int \frac{(\log x)^2}{x}\,\mathrm{d}x$

$t = \log x$ とおくと， $\mathrm{d}t = \dfrac{\mathrm{d}x}{x}$

$$(\text{与式}) = \int t^2\,\mathrm{d}t = \frac{1}{3}t^3 + C = \underline{\underline{(\log x)^3 + C}}$$

## 6 不定積分/置換積分 $\left(\dfrac{f'(x)}{f(x)}\text{型}\right)$

次の不定積分を解け．

(1) $\displaystyle\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x$

$(x^2+x+1)' = 2x+1$ より, (与式) $= \underline{\underline{\log|x^2+x+1| + C}}$

◀ これは部分分数分解などがパッとできない．そのようなときは微分をまず疑う．

(2) $\displaystyle\int \frac{1}{\tan x}\,\mathrm{d}x$

$$(\text{与式}) = \int \frac{\cos x}{\sin x}\,\mathrm{d}x$$

$(\sin x)' = \cos x$ より,

$$= \underline{\underline{\log|\sin x| + C}}$$

(3) $\displaystyle\int \frac{\sin x}{1+\cos x}\,\mathrm{d}x$

$(1+\cos x)' = -\sin x$ より, (与式) $= \underline{\underline{-\log|1+\cos x| + C}}$

(4) $\displaystyle\int \frac{e^x}{e^x+1}\,\mathrm{d}x$

$(e^x+1)' = e^x$ より, (与式) $= \underline{\underline{\log|e^x+1| + C}}$

## 7 不定積分/置換積分ステップアップ

次の不定積分を解け．

(1) $\displaystyle\int (x^3+x)\sqrt{1+x^2}\,\mathrm{d}x$

$$(\text{与式}) = \int x(x^2+1)\sqrt{1+x^2}\,\mathrm{d}x$$

$t = 1 + x^2$ とおくと，$\mathrm{d}t = 2x\,\mathrm{d}x$ だから

$$= \int t\sqrt{t}\cdot\frac{1}{2}\,\mathrm{d}t = \frac{1}{2}\int t^{\frac{3}{2}}\,\mathrm{d}t$$
$$= \frac{1}{2}\cdot\frac{2}{5}t^{\frac{5}{2}} + C = \underline{\underline{\frac{1}{5}\left(1+x^2\right)^{\frac{5}{2}} + C}}$$

(2) $\displaystyle\int \frac{\mathrm{d}x}{e^x+1}$

$t = e^x$ とおくと $\mathrm{d}t = e^x\,\mathrm{d}x$ なので

$$(\text{与式}) = \int \frac{1}{t+1}\cdot\frac{\mathrm{d}t}{t} = \int\left(\frac{1}{t} - \frac{1}{t+1}\right)\mathrm{d}t$$
$$= \log|t| - \log|t+1| + C = \log|e^x| - \log|e^x+1| + C$$
$$= \underline{\underline{x - \log(e^x+1) + C}}$$

◀ $e^x > 0$ より，$\log|e^x| = \log e^x = x$

(3) $\displaystyle\int \frac{e^{2x}}{(e^x+1)^2}\,\mathrm{d}x$

$e^x + 1 = t$ とおくと，$e^x\cdot\mathrm{d}x = \mathrm{d}t$

$$\int\frac{(t-1)^2}{t^2}\,\mathrm{d}t = \int\left(1 - \frac{2}{t} + \frac{1}{t^2}\right)\mathrm{d}t$$
$$= t - \frac{1}{2}\log|t| - \frac{1}{t} + C$$
$$= e^x + 1 - \frac{1}{2}\log(e^x+1) - \frac{1}{e^x+1} + C$$
$$= \underline{\underline{-\frac{1}{2}\log(e^x+1) - \frac{e^{2x}+2e^x}{e^x+1} + C}}$$

(4) $\displaystyle\int \tan^4 x\,\mathrm{d}x$

$$(\text{与式}) = \int \tan^2 x\tan^2 x\,\mathrm{d}x$$
$$= \int \tan^2 x\left(\frac{1}{\cos^2 x} - 1\right)\mathrm{d}x = \int\left(\frac{\tan^2 x}{\cos^2 x} - \tan^2 x\right)\mathrm{d}x$$
$$= \int\left(\frac{\tan^2 x}{\cos^2 x} - \left(\frac{1}{\cos^2 x} - 1\right)\right)\mathrm{d}x$$
$$= \int\frac{\tan^2 x}{\cos^2 x}\,\mathrm{d}x - (\tan x - x)$$

$t = \tan x$ とおくと $\mathrm{d}t = \dfrac{1}{\cos^2 x}\,\mathrm{d}x$ なので

$$= \int t^2\,\mathrm{d}t - (\tan x - x)$$
$$= \frac{1}{3}t^3 - \tan x + x + C$$
$$= \underline{\underline{\frac{1}{3}\tan^3 x - \tan x + x + C}}$$

別解

$t = \tan x$ とすると，$\mathrm{d}t = \dfrac{1}{\cos^2 x}\,\mathrm{d}x = (1+\tan^2 x)\,\mathrm{d}x = (1+t^2)\,\mathrm{d}x$

$$(\text{与式}) = \int t^4 \cdot \frac{\mathrm{d}t}{1+t^2}$$
$$= \int \left(t^2 - 1 + \frac{1}{1+t^2}\right)\mathrm{d}t$$
$$= \frac{t^3}{3} - t + \int \frac{1}{1+\tan x^2}\cdot(1+\tan^2 x)\,\mathrm{d}x$$
$$= \underline{\underline{\frac{\tan^3 x}{3} - \tan x + x + C}}$$

◀ 第 3 項の積分について，置換をもとに戻した.($t$ を $x$ に置換した.)

(5) $\displaystyle\int \frac{x}{1-\cos x}\,\mathrm{d}x$

2 倍角の公式を用いて

$$(\text{与式}) = \int \frac{x}{2\sin^2 \frac{x}{2}}\,\mathrm{d}x = \int x \cdot \left(\frac{1}{\tan \frac{x}{2}}\right)'\,\mathrm{d}x$$
$$= -\frac{x}{\tan \frac{x}{2}} + \int \frac{1}{\tan \frac{x}{2}}\,\mathrm{d}x$$

◀ 部分積分を用いた.

$$= -\frac{x}{\tan \frac{x}{2}} + \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}\,\mathrm{d}x$$

$t = \sin\dfrac{x}{2}$ とおくと $\mathrm{d}t = \dfrac{1}{2}\cos\dfrac{x}{2}\,\mathrm{d}x$ なので

$$
\begin{aligned}
&= -\frac{x}{\tan\frac{x}{2}} + 2\int\frac{\mathrm{d}t}{t} \\
&= -\frac{x}{\tan\frac{x}{2}} + 2\log|t| + C \\
&= \underline{\underline{-\frac{x}{\tan\frac{x}{2}} + 2\log\left|\sin\frac{x}{2}\right| + C}}
\end{aligned}
$$

(6) $\displaystyle\int\frac{1}{1-\sin x}\,\mathrm{d}x$

$$
\begin{aligned}
(\text{与式}) &= \int\frac{1+\sin x}{1-\sin^2 x}\,\mathrm{d}x = \int\left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right)\mathrm{d}x \\
&= \tan x + \int\frac{\sin x}{\cos^2 x}\,\mathrm{d}x
\end{aligned}
$$

$t = \cos x$ とおくと $\mathrm{d}t = -\sin x\,\mathrm{d}x$ なので

$$
\begin{aligned}
&= \tan x - \int\frac{\mathrm{d}t}{t^2} = \tan x + \frac{1}{t} + C \\
&= \underline{\underline{\tan x + \frac{1}{\cos x} + C}}
\end{aligned}
$$

(7) $\displaystyle\int\sqrt{1+2\sqrt{x}}\,\mathrm{d}x$

$t = \sqrt{1+2\sqrt{x}}$ とおくと，$x = \left(\dfrac{t^2-1}{2}\right)^2$

$\mathrm{d}x = 2\cdot\left(\dfrac{t^2-1}{2}\right)\cdot\dfrac{2t}{2}\,\mathrm{d}t = t(t^2-1)\,\mathrm{d}t$

◀ ルートの中身が複雑なときは，ルートごと置換するとうまくいくことが多い．

◀ $x$ について解いてから，微分したほうが楽．

$$
\begin{aligned}
(\text{与式}) &= \int t\cdot t(t^2-1)\,\mathrm{d}t = \int(t^4-t^2)\,\mathrm{d}t \\
&= \frac{t^5}{5} - \frac{t^3}{3} + C = \frac{t^3}{15}\left(3t^2-5\right) + C \\
&= \frac{(1+2\sqrt{x})\sqrt{1+2\sqrt{x}}}{15}\left\{3\left(1+2\sqrt{x}\right)-5\right\} + C \\
&= \frac{2\left(3\sqrt{x}-1\right)\left(1+2\sqrt{x}\right)\sqrt{1+2\sqrt{x}}}{15} + C
\end{aligned}
$$

(8) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{2x^2-4x+3}}$

$$(\text{与式}) = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}x}{\sqrt{(x-1)^2+\frac{1}{2}}}$$

$t = x - 1 + \sqrt{(x-1)^2+\frac{1}{2}}$ とおくと

◀ この置換は知らないと思いつかない．問題で置換を指定されることも多々ある．

$$\mathrm{d}t = \frac{x-1+\sqrt{(x-1)^2+\frac{1}{2}}}{\sqrt{(x-1)^2+\frac{1}{2}}}\,\mathrm{d}x \iff \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{\sqrt{(x-1)^2+\frac{1}{2}}}$$ なので

$$= \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}t}{t} = \frac{1}{\sqrt{2}}\log|t| + C$$
$$= \frac{1}{\sqrt{2}}\log\left(x-1+\sqrt{(x-1)^2+\frac{1}{2}}\right) + C$$
$$= \underline{\underline{\frac{1}{\sqrt{2}}\log\left(x-1+\sqrt{x^2-2x+\frac{3}{2}}\right) + C}}$$

# 8 積分/部分積分その 1

次の不定積分を求めよ．

(1) $\int x\sin(3x-1)\ \mathrm{d}x$

$$= \int x\{-\cos(3x-1)\}'\ \mathrm{d}x$$
$$= x\cdot\left\{-\frac{1}{3}\cos(3x-1)\right\} - \int x'\cdot\left\{-\frac{1}{3}\cos(3x-1)\right\}\ \mathrm{d}x$$
$$= \underline{\underline{-\frac{x}{3}\cos(3x-1) + \frac{1}{9}\sin(3x-1) + C}}$$

(2) $\int 2x\cos(x+1)\ \mathrm{d}x$

$$= \int 2x(\sin(x+1))'\ \mathrm{d}x = 2x\sin(x+1) - \int (2x)'\cdot\sin(x+1)\ \mathrm{d}x$$
$$= 2x\sin(x+1) - \int 2\sin(x+1)\ \mathrm{d}x$$
$$= \underline{\underline{2x\sin(x+1) + 2\cos(x+1) + C}}$$

(3) $\int xe^{2x-1}\ \mathrm{d}x$

$$= \int x\left(\frac{1}{2}e^{2x-1}\right)'\ \mathrm{d}x = x\cdot\frac{1}{2}e^{2x-1} - \int x'\cdot\frac{1}{2}e^{2x-1}\ \mathrm{d}x$$

$$= x \cdot \frac{1}{2}e^{2x-1} - \int \frac{1}{2}e^{2x-1}\,\mathrm{d}x = \underline{\underline{\frac{x}{2}e^{2x-1} - \frac{1}{4}e^{2x-1} + C}}$$

(4) $\displaystyle\int (2x+3)\,e^{2x-5}\ \mathrm{d}x$

$$= \int (2x+3)\left(\frac{1}{2}e^{2x-5}\right)'\ \mathrm{d}x$$
$$= (2x+3)\cdot\frac{1}{2}e^{2x-5} - \int (2x+3)'\cdot\frac{1}{2}e^{2x-5}\ \mathrm{d}x$$
$$= (2x+3)\cdot\frac{1}{2}e^{2x-5} - \int e^{2x-5}\ \mathrm{d}x$$
$$= \underline{\underline{\frac{2x+3}{2}e^{2x-5} - \frac{1}{2}e^{2x-5} + C}}$$

(5) $\displaystyle\int (3x+7)\sin(x+4)\ \mathrm{d}x$

$$= \int (3x+7)\left(-\cos(x+4)'\right)\ \mathrm{d}x$$
$$= (3x+7)\cdot\{-\cos(x+4)\} - \int (3x+7)'\cdot\{-\cos(x+4)\}\ \mathrm{d}x$$
$$= \underline{\underline{(-3x-7)\cos(x+4) + 3\sin(x+4) + C}}$$

(6) $\displaystyle\int (2x-5)\cos(2x-1)\ \mathrm{d}x$

$$= \int (2x-5)\left\{\frac{1}{2}\sin(2x-1)\right\}'\ \mathrm{d}x$$
$$= (2x-5)\cdot\frac{1}{2}\sin(2x-1) - \int (2x-5)'\cdot\frac{1}{2}\sin(2x-1)\ \mathrm{d}x$$
$$= \underline{\underline{\frac{2x-5}{2}\sin(2x-1) + \frac{1}{2}\cos(2x-1) + C}}$$

(7) $\displaystyle\int xe^{x+2}\ \mathrm{d}x$

$$= x\cdot e^{x+2} - \int 1\cdot e^{x+2}\ \mathrm{d}x = \underline{\underline{xe^{x+2} - e^{x+2} + C}}$$

(8) $\displaystyle\int x2^{2x-1}\ \mathrm{d}x$

$$= \int x\left\{2^{2x-1}\right\}'\ \mathrm{d}x$$
$$= x\cdot\frac{2^{2x-1}}{2\log 2} - \int (x)'\cdot\frac{2^{2x-1}}{2\log 2}\ \mathrm{d}x = \underline{\underline{\frac{x2^{2x-1}}{2\log 2} - \frac{2^{2x-1}}{4(\log 2)^2} + C}}$$

(9) $\displaystyle\int \frac{2x}{\cos^2 x}\,\mathrm{d}x$

$$= \int 2x(\tan x)'\,\mathrm{d}x$$
$$= 2x\cdot\tan x - \int (2x)'\cdot\tan x\ \mathrm{d}x = 2x\cdot\tan x - \int 2\frac{(-\cos x)'}{\cos x}\ \mathrm{d}x$$
$$= 2x\cdot\tan x + 2\int\frac{(\cos x)'}{\cos x}\ \mathrm{d}x = \underline{\underline{2x\tan x + 2\log|\cos x| + C}}$$

## 9　積分/部分積分その2

次の不定積分を求めよ．

(1) $\displaystyle\int x^2\cos x\,\mathrm{d}x$

$= \underline{\underline{x^2\sin x + 2x\cos x - 2\sin x + C}}$

(2) $\displaystyle\int \log\frac{x-1}{x+1}\,\mathrm{d}x$

$\displaystyle(\text{与式}) = \int\{\log(x-1)+\log(x+1)\}\ \mathrm{d}x$
$= \underline{\underline{(x-1)\log|x-1| - (x+1)\log|x+1| + C}}$

(3) $\displaystyle\int(\log x)^2\,\mathrm{d}x$

$= \underline{\underline{x\left\{(\log x)^2 - 2\log x + 2\right\} + C}}$

## 10　積分/対数関数

次の不定積分を求めよ．

(1) $\displaystyle\int \log(x+3)\ \mathrm{d}x$
$\displaystyle= (x+3)\log(x+3) - \int(x+3)\cdot\frac{1}{x+3}\ \mathrm{d}x$
$= \underline{\underline{(x+3)\log(x+3) - x + C}}$

◀ 解答ではそのままやるが, $x+3 = t$ と置換したほうがやりやすいかもしれない. $\mathrm{d}x = \mathrm{d}t$

(2) $\displaystyle\int \log(x+4)\ \mathrm{d}x\ = (x+4)\log(x+4) - \int(x+4)\cdot\frac{1}{x+4}\ \mathrm{d}x$
$= \underline{\underline{(x+4)\log(x+4) - x + C}}$

(3) $\displaystyle\int \log 7x\ \mathrm{d}x\ = x\log 7x - \int x\cdot\frac{7}{7x}\ \mathrm{d}x$
$= \underline{\underline{x\log 7x - x + C}}$

(4) $\displaystyle\int \log(x+5)\ \mathrm{d}x\ = (x+5)\log(x+5) - \int(x+5)\cdot\frac{1}{x+5}\ \mathrm{d}x$
$= \underline{\underline{(x+5)\log(x+5) - x + C}}$

(5) $\displaystyle\int \log 3x\ \mathrm{d}x\ = x\log 3x - \int x\cdot\frac{3}{3x}\ \mathrm{d}x$
$= \underline{\underline{x\log 3x - x + C}}$

(6) $\displaystyle\int \log(2x-3)\ \mathrm{d}x$

$2x-3=t$ と置換する. このとき $2\,\mathrm{d}x=\ \mathrm{d}t$

$$
\begin{aligned}
(\text{与式}) &= \int \log t\left(\frac{1}{2}\,\mathrm{d}t\right)\\
&= \frac{1}{2}\left(t\log t-\int\frac{t}{t}\,\mathrm{d}t\right)\\
&= \frac{1}{2}\{(2x-3)\log(2x-3)-(2x-3)+C\}\\
&= \underline{\underline{\ \frac{2x-3}{2}\{\log(2x-3)-1\}+C\ }}
\end{aligned}
$$

# 11　積分/部分積分ステップアップ

次の不定積分を求めよ.

(1) $\displaystyle\int e^x\cos x\,\mathrm{d}x$

$I=\displaystyle\int e^x\cos x\,\mathrm{d}x$ とおく.

◀ 指数×三角関数のときは, 同じ形が得られるまで部分積分を行う.

$$
\begin{aligned}
I &= \int e^{x\prime}\cos x\,\mathrm{d}x\\
&= e^x\cos x-\int e^x(-\sin x)\,\mathrm{d}x\\
&= e^x\cos x+\int e^{x\prime}\sin x\,\mathrm{d}x\\
&= e^x\cos x+e^x\sin x+\int e^x\cos x\,\mathrm{d}x\\
&= e^x\cos x+e^x\sin x+I\\
\Leftrightarrow 2I &= e^x\cos x+e^x\sin x+C\\
\therefore\ \ I &= \underline{\underline{\frac{1}{2}(e^x\cos x+e^x\sin x+C)}}
\end{aligned}
$$

(2) $\displaystyle\int e^{-x}\sin 2x\,\mathrm{d}x$

$J = \int e^{-x} \sin 2x \, \mathrm{d}x$ とおく．

$$
\begin{aligned}
J &= \int (-e^{-x})' \sin 2x \, \mathrm{d}x \\
&= -e^{-x} \sin 2x - \int -e^{-x} \cdot 2 \cos 2x \, \mathrm{d}x \\
&= -e^{-x} \sin 2x + 2 \int (-e^{-x})' \cos 2x \, \mathrm{d}x \\
&= -e^{-x} \sin 2x + 2 \left\{ -e^{-x} \cos 2x - \int -e^{-x} \cdot 2(-\sin 2x) \, \mathrm{d}x \right\} \\
&= -e^{-x} \sin 2x - 2e^{-x} \cos 2x - 4 \int e^{-x} \sin 2x \, \mathrm{d}x \\
&= -e^{-x} \sin 2x - 2e^{-x} \cos 2x - 4J \\
\Leftrightarrow 5J &= -e^{-x} \sin 2x - 2e^{-x} \cos 2x + C \\
\therefore J &= \underline{\underline{\frac{1}{5}(-e^{-x} \sin 2x - 2e^{-x} \cos 2x + C)}}
\end{aligned}
$$